Around the homologous sphere of Poincare and its applications

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Topology of a 3-dim manifolds defined by the system of equations

$$|z_1|^2+|z_2|^2+|z_3|^2-1=0, \quad z_1^l+z_2^m+z_3^n=0, \tag{1}$$

where $z_k=x_k+iy_k$ are the complex coordinates, depends from the values of the parameters $l,\ m,\ n$. In the case $l=2,\ \ m=3,\ \ n=5$ the manifold defined by the conditions (1) is a famous homologous sphere of Poincare, which has a set of homologies same with standard $3D$-sphere $|z_1|^2+|z_2|^2=1$, but differs from it by self fundamental group. It has an important applications in various branch of modern algebraic topology (J.Milnor,1968).

In this report will be told how to represent the homologous sphere defined by intersection of the fife-dimensional sphere with singular manifold ($l=2,\ \ m=3,\ \ n=5$)

$$|z_1|^2+|z_2|^2+|z_3|^2=1, \quad z_1^2+z_2^3+z_3^5=0. \tag{2}$$

in the form of an explicit expression for one function between of the four variables $H(x,y,u,v)=0$.

Theorem 1. *In the Eulerian coordinates*

$$z_1=\cos(\theta)\,\mathrm{e}^{-2/3\,i\sqrt{3}\phi},\ z_2=-\sin(\theta)\sin(1/2\,\beta)\,\mathrm{e}^{-1/2\,i\left(\alpha-\delta+4/3\,\sqrt{3}\phi\right)},$$

$$z_3=\sin(\theta)\cos(1/2\,\beta)\,\mathrm{e}^{1/2\,i\left(\alpha+\delta-4/3\,\sqrt{3}\phi\right)}, \tag{3}$$

the equation of the unit five-dimensional sphere is identically satisfied and the equation of the orbifold $z_1^2+z_2^3+z_3^5=0$ *takes the form*

$$(\cos(\theta))^2\,e^{-4/3\,i\sqrt{3}\phi}-\sin(\theta)\sin(1/2\,\beta)e^{-1/2\,i\sqrt{3}\left(\sqrt{3}\alpha-\sqrt{3}\delta+4\,\phi\right)}+$$

$$+\sin(\theta)\sin(1/2\,\beta)e^{-1/2\,i\sqrt{3}\left(\sqrt{3}\alpha-\sqrt{3}\delta+4\,\phi\right)}\,(\cos(1/2\,\beta))^2+$$

$$+\sin(\theta)\sin(1/2\,\beta)e^{-1/2\,i\sqrt{3}\left(\sqrt{3}\alpha-\sqrt{3}\delta+4\,\phi\right)}\,(\cos(\theta))^2-$$

$$-\sin(\theta)\sin(1/2\,\beta)e^{-1/2\,i\sqrt{3}\left(\sqrt{3}\alpha-\sqrt{3}\delta+4\,\phi\right)}\,(\cos(\theta))^2\,(\cos(1/2\,\beta))^2+$$

$$+\sin(\theta)\,(\cos(1/2\,\beta))^5\,e^{5/6\,i\sqrt{3}\left(\sqrt{3}\alpha+\sqrt{3}\delta-4\,\phi\right)}-$$

$$-2\,\sin(\theta)\,(\cos(1/2\,\beta))^5\,e^{5/6\,i\sqrt{3}\left(\sqrt{3}\alpha+\sqrt{3}\delta-4\,\phi\right)}\,(\cos(\theta))^2+$$

$$+\sin(\theta)\,(\cos(1/2\,\beta))^5\,e^{5/6\,i\sqrt{3}\left(\sqrt{3}\alpha+\sqrt{3}\delta-4\,\phi\right)}\,(\cos(\theta))^4=0. \tag{4}$$

Using then the variable χ*, defined by the condition* $e^{5/2\,i\alpha+5/2\,i\delta-10/3\,i\sqrt{3}\phi}-e^{5\,\chi}=0$*, we express the variable* ϕ *as* $\phi=-1/4\,i\,(i\alpha+i\delta-2\,\chi)\sqrt{3}$ *and after separation of the real and imaginary parts of complex equation ((1)), are obtained two equations into the five variables* $\alpha,\ \delta,\ \chi$ *and* $\theta,\ \beta$ *.*

As result of elimination of the variable χ *from both equations is derived equation of homologous sphere of Poincare in the form of one function of the four variables. The equation is the summa of the functions* $\sin()$ *and* $\cos()$ *with linear arguments. It contains more than* 200 *items.*

By analogy can be considered the case of tetrahedral space which corresponds to the intersection of the fife-dimensional sphere with singular manifold ($l = 2,\quad m = 3,\quad n = 4$)

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = 1, \quad z_1^2 + z_2^3 + z_3^4 = 0. \tag{5}$$

and the octahedral space defined by the condition

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = 1, \quad z_1^2 + z_2^3 + z_2 z_3^3 = 0. \tag{6}$$

Proposition 2. *The relation between a four variables $F(x, y, a, b) = 0$ which defines some $3D$-variety can be considered as General Integral of the par of the second order ODE's $f(x, y, y', y'') = 0$ and $g(a, b, b;, b'') = 0$.*

$$\swarrow F(x, y, a, b) = 0 \searrow$$

$$y'' = f(x, y, y') \quad \longleftrightarrow \quad b'' = h(a, b, b')$$

The Liouville-Tresse invariants of both equations with respect to non degenerate transformations of the variables $x = X(u, v), y = Y(u, v)$ or $a = A(p, q), b = B(p, q)$ can be used for the studies of topological properties of the manifold $F(x, y, a, b) = 0$.

Theorem 3. *Spatial homogeneous the first order system of the equations*

$$\begin{aligned}
\frac{d}{ds}x(s) &= 4\,a_0\,z^2 + (4\,a_2\,y + (3\,a_1 - b_2)\,x)\,z + 4\,a_{22}\,y^2 + \\
&\quad + (3\,a_{12} - 2\,b_{22})\,xy + (2\,a_{11} - b_{12})\,x^2, \\
\frac{d}{ds}y(s) &= 4\,b_0\,z^2 + ((3\,b_2 - a_1)\,y + 4\,b_1\,x)\,z + (2\,b_{22} - a_{12})\,y^2 + \\
&\quad + (-2\,a_{11} + 3\,b_{12})\,xy + 4\,b_{11}\,x^2, \\
\frac{d}{ds}z(s) &= (-b_2 - a_1)\,z^2 + ((-2\,b_{22} - a_{12})\,y - b_{12}\,x - 2\,a_{11}\,x)\,z,
\end{aligned} \tag{7}$$

is projective extension of planar polynomial differential systems

$$\frac{dx}{ds} = a_0 + a_1x + a_2y + a_{11}x^2 + a_{12}xy + a_{22}y^2, \quad \frac{dy}{ds} = b_0 + b_1x + b_2y + b_{11}x^2 + b_{12}xy + b_{22}y^2 \tag{8}$$

with the parameters a_i, a_{ij} and b_i, b_{ij}.

After eliminating of the variables ($y(x)$ or $z(x)$ it is reduced to the second-order differential equations $F(x, y, y', y'') = 0$.

As a result of the exclusion of the function y'' from the system

$$F(x, y, y', y'') = 0, \quad \frac{\partial F}{\partial y''} = 0,$$

a first order differential equation is arised $C(x, y, y') = 0$. Through each point M of integral curve $C = Q(x, y)$ passes the integral curve of the equation $F(x, y, y', y') = 0$, for which the point M is the return point of the second type and this allow to study the limit cycles of the system (8).

References

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